otherwise indicated, we will assume that every multivector is an element of $\mathcal{G}_3$. The perceptive reader will be able to identify those instances where such an assumption is unnecessary.

(3.1) (a) Prove that a vector $\mathbf{x}$ is in the $\mathbf{B}$-plane if and only if $\mathbf{xB} = -\mathbf{Bx}$

(b) Prove that $\mathbf{x}' = \mathbf{xB}$ is a vector with the properties $|\mathbf{x}'| = |\mathbf{B}||\mathbf{x}|$ and $\mathbf{x}\cdot\mathbf{x}' = 0$.

(3.2) If $\mathbf{a}$ and $\mathbf{b}$ are vectors in the plane of $\mathbf{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, show that the ratio of $\mathbf{a}\wedge\mathbf{b}$ to $\mathbf{i}$ is equal to the determinant

$$\begin{vmatrix} \mathbf{a}\cdot\boldsymbol{\sigma}_1 & \mathbf{b}\cdot\boldsymbol{\sigma}_1 \\ \mathbf{a}\cdot\boldsymbol{\sigma}_2 & \mathbf{b}\cdot\boldsymbol{\sigma}_2 \end{vmatrix} = \mathbf{a}\cdot\boldsymbol{\sigma}_1\,\mathbf{b}\cdot\boldsymbol{\sigma}_2 - \mathbf{b}\cdot\boldsymbol{\sigma}_1\,\mathbf{a}\cdot\boldsymbol{\sigma}_2$$

(3.3) Prove the following important identities:

$$\mathbf{a}\cdot\mathbf{b} = -i[\mathbf{a}\wedge(i\mathbf{b})].$$

$$\mathbf{b}\times\mathbf{a} \equiv i(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\cdot(i\mathbf{b}) = -(i\mathbf{b})\cdot\mathbf{a}.$$

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = -\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}\mathbf{c} - \mathbf{a}\cdot\mathbf{c}\mathbf{b}.$$

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = i\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \tfrac{1}{2}(\mathbf{abc} - \mathbf{cba}).$$

Note that the first identity expresses the inner product in terms of the outer product and two duality operations. With the help of the remaining identities, any result of conventional vector algebra can easily be derived from the more powerful results for inner and outer products established in Section 2.1.

(3.4) Reexpress the identities of Exercise (1.1) in terms of the dot and cross products alone.

(3.5) Use an identity in Exercise (1.1) to prove that

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\boldsymbol{\sigma}_1 - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}\boldsymbol{\sigma}_2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\boldsymbol{\sigma}_3.$$

where $a_k \equiv \mathbf{a}\cdot\boldsymbol{\sigma}_k$, $b_k \equiv \mathbf{b}\cdot\boldsymbol{\sigma}_k$.

(3.6) From Equation (3.10), show that

$$A^\dagger = \alpha - i\beta + \mathbf{a} - i\mathbf{b}.$$

$$|A|^2 = \alpha^2 + \beta^2 + \mathbf{a}^2 + \mathbf{b}^2.$$

(3.7) The quaternions can be defined as the set of quantities Q of the form

$$Q = Q_0 + Q_1\mathbf{i}_1 + Q_2\mathbf{i}_2 + Q_3\mathbf{i}_3,$$

where the Q_k ($k = 0, 1, 2, 3$) are scalar coefficients and the $\mathbf{i}_k$ satisfy the equations

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = -1.$$

$$\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3 = 1.$$

Show that the bivector basis given by Equations (3.6) has these properties.

Hamilton used the symbols $\mathbf{i}, \mathbf{j}, \mathbf{k}$ instead of $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ and wrote down the famous equations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ijk} = -1.$$

Of what geometrical significance is the difference in sign between this last equation and the corresponding equation above?

(3.8) The expansion of a vector $\mathbf{b}$ in terms of its components $b_k = \mathbf{b}\cdot\boldsymbol{\sigma}_k$ is commonly expressed by any of the notations

$$\mathbf{b} = b_k\boldsymbol{\sigma}_k = \sum_k b_k\boldsymbol{\sigma}_k = \sum_{k=1}^{3} b_k\boldsymbol{\sigma}_k = b_1\boldsymbol{\sigma}_1 + b_2\boldsymbol{\sigma}_2 + b_3\boldsymbol{\sigma}_3.$$

The most abbreviated form $b_k\boldsymbol{\sigma}_k$ employs the so called *summation convention*, which calls for summation over all allowed values of a repeated pair of indicies in a single term. By this convention, the expansion of a bivector $\mathbf{B}$ in terms of components

$$B_{ij} = \boldsymbol{\sigma}_i\cdot\mathbf{B}\cdot\boldsymbol{\sigma}_j = (\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_i)\cdot\mathbf{B} = -B_{ji}$$

can be written

$$\mathbf{B} = \tfrac{1}{2}B_{ij}\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_i = B_{12}\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1 + B_{31}\,\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_3 + B_{23}\boldsymbol{\sigma}_3\wedge\boldsymbol{\sigma}_2.$$

Show that the duality relation

$$\mathbf{B} = i\mathbf{b}$$

can be expressed in terms of components by the equations

$$B_{ij} = \varepsilon_{ijk}b_k,$$

where ε_{ijk} is defined by

$$\varepsilon_{ijk} = \frac{\boldsymbol{\sigma}_i\wedge\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_k}{i} = i^\dagger\boldsymbol{\sigma}_i\wedge\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_k.$$

Note that $\varepsilon_{ijk} = 0$ if any pair of indicies have the same value, then

$$\varepsilon_{ijk} = \varepsilon_{123} = 1 \text{ if } \{i, j, k\} \text{ is an even permutation of } \{1, 2, 3\}.$$
$$\varepsilon_{ijk} = \varepsilon_{321} = -1 \text{ if } \{i, j, k\} \text{ is an odd permutation of } \{1, 2, 3\}.$$

Also prove that

$$\mathbf{a}\times\mathbf{b} = a_ib_j\varepsilon_{ijk}\boldsymbol{\sigma}_k,$$

$$\frac{\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}}{i} = \varepsilon_{ijk}a_ib_jc_k.$$

(3.9) Prove

$$\boldsymbol{\sigma}_k\mathbf{a}\boldsymbol{\sigma}_k = -\mathbf{a}.$$
$$\boldsymbol{\sigma}_k\mathbf{a}\wedge\mathbf{b}\boldsymbol{\sigma}_k = \mathbf{b}\wedge\mathbf{a}.$$
$$\boldsymbol{\sigma}_k\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\boldsymbol{\sigma}_k = 3\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}. \qquad \text{(sum over } k\text{)}$$


A_1
A_2
B_1
C_1
C_2

properties, we shall wait for them to arise in the context of physical problems, and we will be better prepared for this when we have the tools of differential calculus at our disposal.

Our study of analytic geometry has just begun. The study of particle trajectories, which we undertake in the next chapter, is largely analytic geometry in $\mathcal{E}_3$. For those who wish to study the classical analytic geometry in $\mathcal{E}_2$ in more detail, the book of Zwikker (1963) is recommended. He formulates analytic geometry in terms of complex numbers and shows how much this improves on the traditional methods of coordinate geometry. Of course, everything he does is easily reexpressed in the language of geometric algebra, which has all the advantages of complex numbers and more. Indeed, geometric algebra brings further improvements to Zwikker's treatment by enlarging the algebraic system from $\mathcal{G}_2^+$ to $\mathcal{G}_2$, and so introducing the fundamental distinction between vectors and spinors and along with it the concepts of inner and outer products. Most important, geometric algebra provides for the generalization of the geometry in $\mathcal{E}_2$ to $\mathcal{E}_3$. The present book develops all the principles and techniques needed for analytic geometry, but Zwikker's book is a valuable storehouse of particular facts about curves in $\mathcal{E}_2$. Among other things, it includes the remarkable proof that conic sections as defined by (6.34) really are sections of a cone.

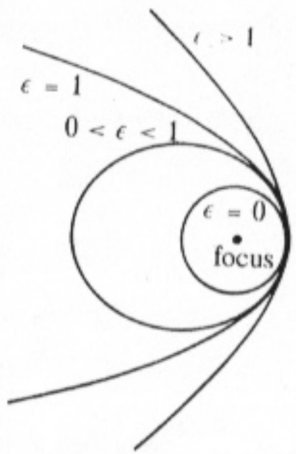


Fig. 6.11. Conics with a common focus and pericenter.

2-6. Exercises

(6.1) From Equation (6.2) derive the following equations for the line $\mathscr{L}$ in terms of rectangular coordinates in $\mathcal{E}_3$:

$$\frac{x_1 - a_1}{u_1} = \frac{x_2 - a_2}{u_2} = \frac{x_3 - a_3}{u_3},$$

where $x_k = \mathbf{x}\cdot\boldsymbol{\sigma}_k$, $a_k = \mathbf{a}\cdot\boldsymbol{\sigma}_k$, $u_k = \mathbf{u}\cdot\boldsymbol{\sigma}_k$.

(6.2) (a) Show that Equation (6.2) is equivalent to the parametric equation

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{u}^{-1}.$$

(b) Describe the solution set $\{\mathbf{x} = \mathbf{x}(t)\}$ of the parametric equation

$$\mathbf{x} = \mathbf{a} + t^2\mathbf{u}$$

for all scalar values of the parameter t.

(6.3) (a) Compute the directance to the line through points $\mathbf{a}$ and $\mathbf{b}$ from the origin.

(b) Compute the directance to this line from an arbitrary point $\mathbf{c}$.

(6.4) Prove the theorem "Three points not on a line determine a plane" by using geometric algebra to derive an equation for the plane from three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(6.5) Describe the solution set $\{x\}$ of the simultaneous equations

$$\mathbf{x}\wedge\mathbf{A} = 0, \qquad \mathbf{x}\wedge\mathbf{B} = 0$$

if $\mathbf{A}$ and $\mathbf{B}$ are noncommuting blades of grade 2.

(6.6) Find the point of intersection of the line $\{\mathbf{x}\}$ determined by the equation $(\mathbf{x} - \mathbf{a})\wedge\mathbf{u} = 0$ with the plane $\{\mathbf{y}\}$ determined by the equation $(\mathbf{y} - \mathbf{b})\wedge\mathbf{B} = 0$. What are the conditions on $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{B}$ that this point exists and is unique?

(6.7) The *directance* from one point set to another can be defined quite generally as the chord of minimum length between points in the two sets, provided there is only one such chord.

Determine the directance $\mathbf{d}$ from a line with direction $\mathbf{u}$ through a point $\mathbf{a}$ to a line with direction $\mathbf{v}$ through a point $\mathbf{b}$. Show that the lines intersect only if $(\mathbf{a} - \mathbf{b})\wedge\mathbf{u}\wedge\mathbf{v} = 0$.

(6.8) Compute the directance from a point $\mathbf{b}$ to the plane $\{\mathbf{x}: (\mathbf{x} - \mathbf{a})\wedge\mathbf{U} = 0\}$.

(6.9) Show that the equation

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

subject to the conditions

$$\alpha + \beta + \gamma = 1 \quad \text{and} \quad \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} \neq 0$$

can be regarded as a parametric equation for a plane. Find a nonparametric equation for this plane.

(6.10) *Ceva's Theorem*: Suppose that concurrent lines from the verticies $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of a triangle divide the opposing sides at $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ (Figure 6.12). Then the division ratios satisfy

$$\left(\frac{\mathbf{a} - \mathbf{c}'}{\mathbf{c}' - \mathbf{b}}\right)\left(\frac{\mathbf{b} - \mathbf{a}'}{\mathbf{a}' - \mathbf{c}}\right)\left(\frac{\mathbf{c} - \mathbf{b}'}{\mathbf{b}' - \mathbf{a}}\right) = 1.$$

Prove by showing that the areas indicated in the figure satisfy

$$\frac{\mathbf{A}_1}{\mathbf{A}_2}\frac{\mathbf{B}_1}{\mathbf{B}_2}\frac{\mathbf{C}_1}{\mathbf{C}_2} = 1.$$

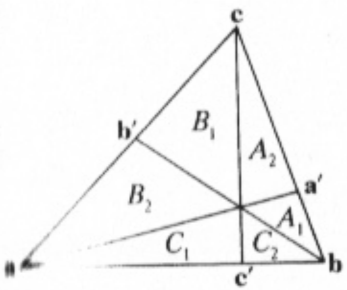
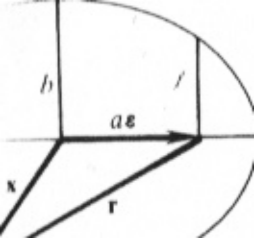


Fig. 6.12. *Ceva's Theorem*.

(6.11) In Figure 6.5 we have the division ratios


ε = 0
focus


b
l
ae
x
r

$$\lambda = \frac{\mathbf{x}-\mathbf{b}}{\mathbf{b}-\mathbf{a}}, \qquad \mu = \frac{\mathbf{c}-\mathbf{0}}{\mathbf{x}-\mathbf{c}}, \qquad \nu = \frac{\mathbf{a}-\mathbf{y}}{\mathbf{y}-\mathbf{0}}.$$

Prove the *theorem of Menelaus*: $\lambda\mu\nu = -1$. Note that the theorem can be interpreted as expressing a relation among intersecting sides of a quadrilateral with verticies $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ or as a property of a transversal cutting the triangle with verticies $\mathbf{0}$, $\mathbf{a}$, $\mathbf{x}$.

(6.12) Prove that three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ lie on a line iff there exist nonzero scalars α, β, γ such that $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$ and $\alpha + \beta + \gamma = 0$.

(6.13) *Desargues' Theorem.* Given two triangles $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$. Then lines through corresponding verticies are concurrent (at a point $\mathbf{s}$) iff lines along corresponding sides intersect at collinear points ($\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$). (Figure 6.13) Note that the triangles need not lie in the same plane.

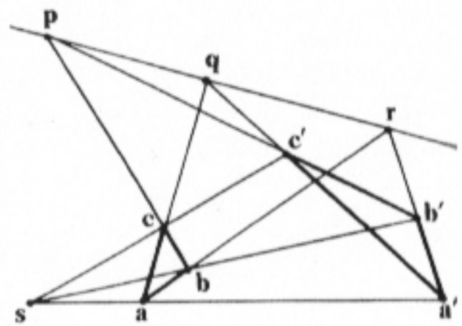


Fig. 6.13. *Desargue's Theorem.*

(6.14) The equation $(\mathbf{x} - \mathbf{b})\cdot\mathbf{u} = 0$ describes a plane in $\mathcal{E}_3$ with normal $\mathbf{u}$. Derive this equation from Equation (6.21).

(6.15) Four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ determine a tetrahedron with directed volume

$$\begin{aligned} V &= \tfrac{1}{6}\,(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a})\wedge(\mathbf{d}-\mathbf{a}) \\ &= \tfrac{1}{6}\,(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d} - \mathbf{c}\wedge\mathbf{d}\wedge\mathbf{a} + \mathbf{d}\wedge\mathbf{a}\wedge\mathbf{b} - \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}). \end{aligned}$$

Use this to determine the equation for a plane through three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(6.16) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the directions of three coplanar lines. The relative directions of the lines are then specified by $\alpha = \mathbf{b}\cdot\mathbf{c}$, $\beta = \mathbf{a}\cdot\mathbf{c}$, $\gamma = \mathbf{a}\cdot\mathbf{b}$. Prove that

$$2\alpha\beta\gamma = \alpha^2 + \beta^2 + \gamma^2 - 1.$$

(6.17) Determine the parametric values λ_1, λ_2 for which the line $\mathbf{x} = \mathbf{x}(\lambda) = \mathbf{a} + \lambda\mathbf{u}$ ($\mathbf{u}^2 = 1$) intersects the circle with equation $\mathbf{x}^2 = r^2$, and show that $\lambda_1\lambda_2 = a^2 - r^2$ for every line through $\mathbf{a}$ which intersects the circle.

(6.18) Show that tangents to the circle of radius r and center at the origin in $\mathcal{E}_2$ which pass through a given point $\mathbf{a}$ intersect the circle at the points.

$$\mathbf{d}_\pm = \left(1 + \frac{\alpha_\pm}{r}\mathbf{i}\right)^{-1}\mathbf{a} = (r - \alpha_\pm\mathbf{i})r\mathbf{a}^{-1},$$

where $\alpha_\pm = \pm(a^2 - r^2)^{1/2}$ and $\mathbf{i}$ is the unit bivector.

(6.19) Find the radius r and center $\mathbf{c}$ of the circle determined by Equation (6.28).

(6.20) Let x and y be rectangular coordinates of the point $\mathbf{x}$. Show that the defining equation (6.35) of a conic is equivalent to the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

for an ellipse and a hyperbola respectively, where

$$a = \frac{\ell}{|1-\varepsilon^2|}, \qquad b^2 = a\ell, \qquad \mathbf{x} = \mathbf{r} + a\boldsymbol{\varepsilon}.$$

The curves and related parameters are shown in Figures 6.14a, b.

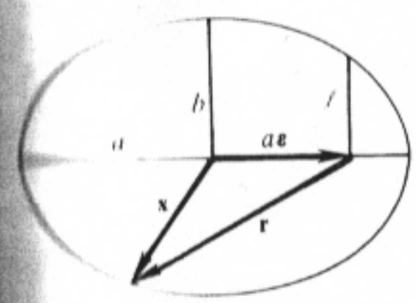


Fig. 6.14a. Ellipse.

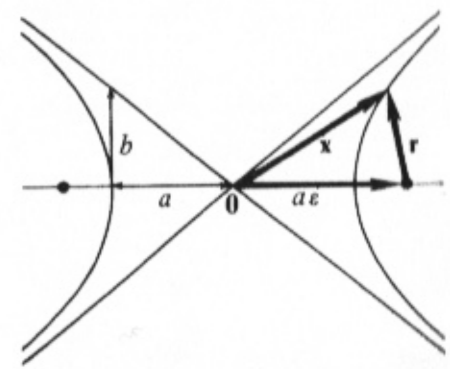


Fig. 6.14b. Hyperbola.

Use the above equations to show that an ellipse has a parametric equation $\mathbf{x} = \mathbf{x}(\phi)$ with the explicit form

$$\mathbf{x} = \mathbf{a}\cos\phi + \mathbf{b}\sin\phi,$$

while a hyperbola has the parametric equation

$$\mathbf{x} = \mathbf{a}\cosh\phi + \mathbf{b}\sinh\phi,$$

where $\mathbf{a}^2 = a^2$, $\mathbf{b}^2 = b^2$ and $\mathbf{a}\cdot\mathbf{b} = 0$.

(6.21) Parametric curves $\mathbf{x} = \mathbf{x}(\lambda)$ of the second order are defined by equation

$$\mathbf{x} = \frac{\mathbf{a}_0 + \mathbf{a}_1\lambda + \mathbf{a}_2\lambda^2}{\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2}.$$

Note that this generalizes the Equation (6.16) for a line. By the change of parameters $\lambda \to \lambda - \alpha_1/2\alpha_2$, this can be reduced to the form

$$\mathbf{x} = \frac{\mathbf{a}_0 + \mathbf{a}_1\lambda + \mathbf{a}_2\lambda^2}{\alpha + \lambda^2}$$

We now aim to show that this equation describes an *ellipse* iff $\alpha > 0$, a ***parabola*** iff $\alpha = 0$, and a ***hyperbola*** iff $\alpha < 0$, where iff means "if and only if". Thus, all conics are second order curves and conversely. Show that

(a) For $\alpha = 1$, the change of parameters $\lambda = \tan \frac{1}{2}\phi$ enables us to put the equation in the form

$$\mathbf{x} = \mathbf{a}\cos\phi + \mathbf{b}\sin\phi + \mathbf{c},$$

which we recognize as a general equation for an ellipse.

(b) For $\alpha = -1$, $\lambda = \tanh \frac{1}{2}\phi$ gives

$$\mathbf{x} = \mathbf{a}\cosh\phi + \mathbf{b}\sinh\phi + \mathbf{c}.$$

(6.22) Solve Equation (6.28) for **x** and put it in the general form given in exercise (6.21).

(6.23) Let $\mathbf{x} = \mathbf{x}(\lambda) = \boldsymbol{\sigma}_1 z(\lambda)$ describe a curve in $\mathcal{E}_2$. Identify and draw diagrams of the curves determined by the following specific forms for the spinor z.

(a) $z = (1 + \mathbf{i}\lambda)^{1/2}$

(b) $z = \dfrac{1}{1 - \mathbf{i}\lambda}$

(c) $z = (1 - \mathbf{i}\lambda)\, e^{\mathbf{i}\lambda}$

(d) $z = (1 - \mathbf{i}\lambda)^{-1/2}$.

(6.24) Describe the solution set $\{\mathbf{x}\}$ in $\mathcal{E}_3$ determined by the following equations. Comment, especially, on the dependence of the solution set on vector parameters **a**, **b**, **c**.

(a) $(\mathbf{a}\cdot\mathbf{x})^2 = \mathbf{x}^2$.

(b) $\mathbf{a}\cdot\mathbf{x} \geqslant |\,\mathbf{x}\,|$.

(c) $\langle(\mathbf{ax})^2\rangle_0 = 0\,; \qquad \langle(\mathbf{ax})^2\rangle_2 = 0$.

(d) $\langle(\mathbf{ax})^2\rangle_0 = 1$.

(e) $\mathbf{x}\cdot\mathbf{a} + (\mathbf{x}\wedge\mathbf{a})^2 = 0$.

(f) $(\mathbf{a}\cdot\mathbf{x})^2 = \mathbf{x}^2 \quad$ and $\quad (\mathbf{x} - \mathbf{c})\cdot\mathbf{b} = 0$.

2-7. Functions of a Scalar Variable

In this section we review some basic concepts of differential and integral calculus to show how they apply to multivector-valued functions.

If to each value of a scalar variable t there corresponds a multivector $F(t)$, then $F = F(t)$ is said to be a (multivector-valued) *function* of t. It is important to distinguish between the function F and the functional value $F(t_0)$, the particular multivector which corresponds to the particular scalar t_0. However, it is often inconvenient to make that distinction explicit in the mathematical notation, so the reader will be left to infer it from the context. Thus, $F(t)$ will denote a functional value if t is understood to be a specific real number, but $F(t)$ will denote a function if no specific value is attributed to t.* Similarly, when the variable t is suppressed, F may indicate a value of the function instead of the function itself. So $F = F(t)$ may refer either to a function or a functional value. It should be understood, also, that the function $F(t)$ is not completely defined until the values of the variable t for which it is defined have been specified. However, the reader will usually be left to infer the allowed values of a variable from the context.

Continuity

The function $F(t)$ is said to be *continuous* at t_0 if

$$\lim_{t\to t_0} |\,F(t) - F(t_0)\,| = 0. \tag{7.1a}$$

We write

$$\lim_{t\to t_0} F(t) = F(t_0), \quad \text{or} \quad F(t) \to F(t_0). \tag{7.1b}$$

The definition of "limit" presumed in (7.1a) is the same as the one introduced in elementary calculus. It applies to multivectors, because we have already introduced an appropriate definition of the "absolute value" $|\,F(t)\,|$, and, in spite of the fact that the geometric product is not commutative, it can be proved that for multivector-valued functions $F(t)$ and $G(t)$ we have the elementary results

$$\lim_{t\to t_0} F(t) + G(t) = \lim_{t\to t_0} F(t) + \lim_{t\to t_0} G(t), \tag{7.2a}$$

$$\lim_{t\to t_0} F(t)G(t) = \lim_{t\to t_0} F(t) \lim_{t\to t_0} G(t). \tag{7.2b}$$

To this we can add the (almost trivial) result

$$\langle \lim_{t\to t_0} F(t)\rangle_k = \lim_{t\to t_0} \langle F(t)\rangle_k. \tag{7.2c}$$

It follows, then, that the function F is continuous if its k-vector parts $\langle F\rangle_k$ are continuous functions.

*It may be noted that a variable is itself a function. Given a set, the *variable* on that set is just the identity function, namely that function which associates each element of the set with itself. So $F(t)$ can be interpreted as the composite of two functions F and t.