Geometric Algebra: Formulae

We assume here that $a, b, c \ldots$ are vectors, $A, B, C \ldots$ are multivectors, $A_r, B_s, C_t \ldots$ are blades or homogeneous multivectors, π is a permutation of $\{1, 2, \ldots, n\}$ and $|\pi|$ is the number of transpositions to get to the identity from π.

- Basic

  1.

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) \tag{1}$$

$$= a \cdot b + a \wedge b \tag{2}$$

$$A_r B_s = \sum_{k=0}^{m} < A_r B_s >_{r+s-2k} \text{ where } m = \tfrac{1}{2}(r + s - |r - s|) \tag{3}$$

ordering r-vector parts

$$< A^\dagger >_r = < A >_r^\dagger = (-1)^{r(r-1)/2} < A >_r \tag{4}$$

$$< AB >_r = (-1)^{r(r-1)/2} < B^\dagger A^\dagger >_r \tag{5}$$

$$< A_r B_s >_r = < B_s^\dagger A_r >_r = (-1)^{s(s-1)/2} < B_s A_r >_r \tag{6}$$

$$< A B_r C >_r = < C^\dagger B_r A^\dagger >_r \tag{7}$$

$$< A_r B_s C_t >_q = (-1)^\epsilon < C_t B_s A_r >_q \tag{8}$$

inner and outer products

$$a_1 \wedge \ldots \wedge a_n = \frac{1}{n!} \sum_{\pi} (-1)^{|\pi|} a_{\pi(1)} a_{\pi(2)} \ldots a_{\pi(n)} \tag{9}$$

$$A_r \cdot B_s = < AB >_{|r-s|} \tag{10}$$

$$A_r \wedge B_s = < AB >_{r+s} \tag{11}$$

$$a \cdot A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) \tag{12}$$

$$a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) \tag{13}$$

$$aA = a \cdot A + a \wedge A \tag{14}$$

$$A_r \cdot (B_s \cdot C_t) = (A_r \wedge B_s) \cdot C_t \text{ for } r+s \leq t \tag{15}$$

$$A_r \cdot (B_s \cdot C_t) = (A_r \cdot B_s) \cdot C_t \text{ for } r+t \leq s \tag{16}$$

$$a \cdot (A_r B) = a \cdot A_r B + (-1)^r A_r a \cdot B \tag{17}$$

$$= a \wedge A_r B - (-1)^r A_r a \wedge B \tag{18}$$

$$a \wedge (A_r B) = a \wedge A_r B - (-1)^r A_r a \cdot B \tag{19}$$

$$= a \cdot A_r B + (-1)^r A_r a \wedge B \tag{20}$$

Laplace expansions

$$a \cdot (a_1 a_2 \ldots a_r) = \sum_{k=1}^{r} (-1)^{k+1} a \cdot a_k (a_1 \ldots \check{a}_k \ldots a_r) \tag{21}$$

$$a \cdot A_r = \sum_{k=1}^{r} a \cdot a_k a_k^{-1} A_r \text{ if } A_r = a_1 \ldots a_r \tag{22}$$

$$B_r \cdot (a_1 \wedge \ldots \wedge a_n) = \frac{1}{r!} \sum_{\pi} (-1)^{|\pi|} B_r \cdot (a_{\pi(1)} \wedge \ldots \wedge a_{\pi(r)}) \; a_{\pi(r+1)} \wedge \ldots \wedge a_{\pi(n)} \tag{23}$$

$$= \sum_{j_1 < j_2 < \ldots < j_r} \pm B_r \cdot (a_{j_1} \ldots \wedge a_{j_r})(a_{j_{r+1}} \ldots \wedge a_{j_n}) \tag{24}$$

$$(b_r \wedge b_{r-1} \wedge \ldots \wedge b_1) \cdot (a_1 \wedge \ldots \wedge a_r) = \det(b_j \cdot a_k)_{jk} \tag{25}$$

Linear Algebra

$$f(a_1 \wedge a_2 \wedge \ldots \wedge a_r) = f(a_1) \wedge f(a_2) \ldots \wedge f(a_r) \quad (26)$$

$$= (\det f) a_1 \wedge a_2 \wedge \ldots \wedge a_r \quad (27)$$

$$(a_r \wedge \ldots \wedge a_2 \wedge a_1) \cdot (b_1 \wedge \ldots \wedge a_r) = \det(a_i \cdot b_j) \quad (28)$$

$$\bar{f}(x) \cdot y = x \cdot f(y) \text{ adjoint or transpose} \quad (29)$$

$$f^{-1}(y) = \frac{1}{\det f} \bar{f}(yi) i^{-1} \text{ with } i = \sigma_1 \ldots \sigma_n \quad (30)$$

Frames: For $A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r \neq 0$ for $r = 1, 2, \ldots, n$. $A_0 = 1$.

$$c_r = A_{r-1}^{\dagger} A_r \text{ are } \perp \text{ for } r = 1, \ldots, n. \text{ i.e. Gram-Schmidt} \quad (31)$$

$$a^r = (-1)^{r-1} a_1 \wedge \ldots \check{a}_r \wedge a_{r+1} \ldots \wedge a_n A_n^{-1} \text{ reciprocal basis.} \quad (32)$$

$$a^j \cdot a_k = \delta_k^j \quad (33)$$