Lattice Theory, Probability, and Relevance

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Abstract

In this tutorial, I will introduce probability theory from the perspective of lattice theory where the fundamental notion is based on a partial ordering of logical statements ordered by implication. This construction forms a Boolean lattice, which is related to the fact that the corresponding logical operations form a Boolean algebra.

I generalize the concept of logical implication to degrees of implication by generalizing the Zeta function of the lattice. The rules of probability theory come out naturally as a set of constraint equations. Through this construction we are able to neatly link order to structure to algebra to calculus. The meaning of probability is inherited from the meaning of the ordering relation, implication, rather than being imposed in an ad hoc manner at the start.

Last, I will briefly introduce a definition of questions introduced by Cox, and show how the lattice of questions under the natural ordering is dual to the lattice of assertions—not by an isomorphic symmetry, but rather by a symmetry known as Birkhoff's Representation Theorem. The result is not a Boolean algebra, as Cox, Belnap and others erroneously believed, but instead a far-richer algebra that presents some interesting challenges for those interested in deriving measures, such as relevance, on that space.