Sourav Chatterjee 2004 Trick

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February 26, 2020

Assume (X, X') μ-exchangeable variables, i.e.

$$\mu\{X \in A, X' \in B\} = \mu\{X' \in A, X \in B\}$$

every function splits as,

$$h(X) = \frac{1}{2}[h(X) + h(X')] + \frac{1}{2}[h(X) - h(X')]$$

i.e., $h = h^+ + h^- = \text{sym.} + \text{antisym.}$ and we have,

$$\mu h = \int h(x)\, d\mu = \mu h^+ \text{ since } \mu h^- = 0$$

Given,

$$F(X, X') = -F(X', X) \text{ antisymmetric}$$

Let

$$f(X) = \mu_X F \ = E[F(X, X')|X]$$

and thus,

$$\mu h f = \mu h F = \mu h^- F \text{ since } \mu h^+ F = 0.$$

Theorem (Chatterjee 2004)**.** *Just set $h = f$ above to get,*

1. *The Trick!*

$$\mu f^2 = \mu f^- F$$

2. *Hoeffding++*

$$\mu_X |f^- F| \leq C \Rightarrow \mu e^{\theta f} \leq e^{C\theta^2/2} \text{ for all } \theta \in R$$

and

$$\mu\{|f| > t\} \leq 2e^{-t^2/2C} \text{ for all } t \geq 0$$

Proof. If follows the proof of Hoeffding's theorem very closely. Let $\theta \in R$ and define $h(X) = e^{\theta f(X)}$. The m.g.f. of $f(X)$ is,

$$m(\theta) = \mu e^{\theta f} = \mu h$$

and The Trick gives,

$$|m'(\theta)| = |\mu h f| = |\mu h^- F| \leq \mu |h^- F|$$

The convexity of $x \to e^x$ produces the well known inequality,

$$\left| \frac{e^x - e^y}{x - y} \right| \leq \frac{1}{2}(e^x + e^y)$$

that translates into,

$$|h^-| \leq h^+ |\theta f^-|$$

thus,

$$|m'(\theta)| \leq |\theta| \, \mu \, h^+ |f^- F| = |\theta| \, \mu \, h |f^- F|$$

since $h^- |f^- F|$ is antisymmetric. Hence,

$$|m'(\theta)| \leq |\theta| \, C m(\theta)$$

Just solve for $m(\theta)$ and optimize (just like in Hoeffding's theorem) to get,

$$m(\theta) \leq e^{C\theta^2/2}$$

and,

$$\mu\{|f| > t\} \leq 2e^{-t^2/2C}$$

□

Eventhough the proof is sort of trivial (which is Great!), the freedom of choosing the pair (X, X') of exchengeable variables and the antisymmetric function F, makes Chatterjee's beautiful trick invaluable in situations involving functions of dependent variables e.g. Hamiltonians for ferromagnetic models.